

EDGE STATES IN GAUGE THEORIES: THEORY, INTERPRETATIONS AND PREDICTIONS

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Abstract

Gauge theories on manifolds with spatial boundaries are studied. It is shown that observables localized at the boundaries (edge observables) can occur in such models irrespective of the dimensionality of spacetime. The intimate connection of these observables to charge fractionation, vertex operators and topological field theories is described. The edge observables, however, may or may not exist as well-defined operators in a fully quantized theory depending on the boundary conditions imposed on the fields and their momenta. The latter are obtained by requiring the Hamiltonian of the theory to be self-adjoint and positive definite. We show that these boundary conditions can also have nice physical interpretations in terms of certain experimental parameters such as the penetration depth of the electromagnetic field in a surrounding superconducting medium. The dependence of the spectrum on one such parameter is explicitly exhibited for the Higgs model on a spatial disc in its London limit. It should be possible to test such dependences experimentally, the above Higgs model for example being a model for a superconductor. Boundary conditions for the 3+1 dimensional BF system confined to a spatial ball are studied. Their physical meaning is clarified and their influence on the edge states of this system (known to exist under certain conditions) is discussed. It is pointed out that edge states occur for topological solitons of gauge theories such as the 't Hooft-Polyakov monopoles.

1 Introduction

The study of gauge fields on spatial manifolds with boundaries has attracted attention in recent times. A strong incentive for these investigations comes from attempts to model Quantum Hall Effect (QHE) where edge states localized at spatial boundaries are known to exist and to be of fundamental physical significance [1]. A second reason for these investigations is the remarkable relation of these states to conformal field theories (CFT's) and the possibility thereby suggested of a three-dimensional approach to these two-dimensional theories [2, 3].

In previous work, we have examined Chern-Simons (CS) [3], Maxwell-Chern-Simons (MCS) [4] and Higgs field dynamics [5] with disk as spatial manifold and reproduced many previously known results on edge states. It was also established that much of the previous work is limited in scope and fails to exhaust all available physical possibilities. This is because quantization depends on certain boundary conditions (BC's). The latter, for a Hamiltonian bounded below, and with suitable additional assumptions, can be parametrized by a non-negative parameter λ for the CS and MCS cases, and by two nonnegative parameters λ and μ for the Higgs case. All work we know of prior to [4] had tacitly assumed that $\lambda = 0$. In [4], we had also announced an interpretation of λ : if the disk is surrounded by a superconductor, then $1/\lambda$ is proportional to the penetration depth into its ambient medium. That is, λ is proportional to the vector meson mass in the surrounding medium.

In this paper, we will explain edge states using a few elementary examples and then will establish the interpretation of λ stated above. [We do not however have a good interpretation of μ .] With this meaning at hand, it is easy to imagine experiments to observe the effects of λ , as we will see. Further, we will extend our previous work on edge states of the BF system in three spatial dimensions [6] by enlarging the boundary

conditions contemplated. They too are characterized by parameters like λ and μ with properties similar to those in two dimensions.

Section 2 initiates our considerations by introducing edge states using Maxwell's theory in $1 + 1$ spacetime, the spatial slice being the interval $[-\frac{L}{2}, \frac{L}{2}]$. Although the quantum physics of this model is simple, it is nevertheless rich enough to display several phenomena of novelty and interest. Thus we will see that there are charged edge states localized at $\pm \frac{L}{2}$ and that they are created by “vertex” operators (or “Wilson” integrals). The Coulomb force between the charges comes out correctly in this approach. We will also encounter fractionation of charge by a mechanism first discovered in monopole theory by Witten [7, 8]. Just as for dyons, this phenomenon is related to the existence of a “ θ -angle” here as well. As a final result of this section, we will find that there is a topological field theory which describes these states. This transparent model in this manner nicely illustrates important ideas found in more complex field theories.

Section 3 continues the preceding discussion to $2 + 1$ dimensional spacetime with a disc or an annulus as the spatial slice. Just as in Section 2, we find here also charge fractionation, vertex operators and a topological field theory for edge states, the latter being a CS theory for a generic MCS Lagrangian. In this section we also summarize the results found in [5] for a gauged Higgs theory and briefly discuss why this theory does not have edge states.

The discussion of the last two Sections tacitly assumes the parameters λ and μ mentioned above to be zero and is therefore incomplete. We had previously discussed the origin of these parameters from the characterization of certain boundary conditions in the mathematical analysis, and also the quantization of our Lagrangians for their generic values [4, 5]. In Section 4, we summarize these boundary conditions for convenience while we establish the physical meaning of λ in Section 5. From this one sees that it is possible to vary λ in an experimental arrangement. Now physical properties are sensitive to their

values and affected by their variations. The energy spectrum for example is affected by λ and μ while edge states, existing in the MCS theory for $\lambda = 0$, get delocalized when λ is changed from this critical number. It seems for these reasons possible to test our work experimentally. With this possibility in mind, we also quote estimates for the dependence of the energy spectrum on λ in Section 6, using our work on the gauged Higgs system for generic λ (and $\mu = 0$) [5] as a guide. This work incidentally not only quantizes the gauged Higgs system for any λ and μ but also argues that it does not admit U gauge for finite geometry and general λ and μ . The latter is established by showing that the gauged Higgs system in the London limit is not equivalent to a massive vector meson theory for this geometry.

Section 7 is the final one and concerns spatial manifolds with boundaries in $3 + 1$ dimensions. It recalls previous work [6] on the BF system showing that edge states can occur in $3+1$ dimensions as well. States similar to the edge states of vortices remarked upon in Sections 3 and 4 are shown to exist also for the 't Hooft-Polyakov monopoles. The possibility of BC's depending on parameters like λ and μ , with properties similar to those in $3+1$ dimensions, is also shown. The edge states spread out now too when λ deviates from zero. They should be observable by the dependence of their properties, such as energies, on these parameters.

2 Edge Observables in $1 + 1$ dimensions

We here consider abelian Maxwell's theory on $[-\frac{L}{2}, \frac{L}{2}] \times \mathbf{R}^1$, L being the spatial length and \mathbf{R}^1 accounting for time. It is defined by the Hamiltonian

$$H = \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} E(x)^2 dx , \quad (2.1)$$

the equal time commutators

$$[A_1(x), A_1(y)] = [E(x), E(y)] = 0 , \quad (2.2)$$

$$[A_1(x), E(y)] = i\delta(x - y) \quad (2.3)$$

and the Gauss law constraint

$$\frac{\partial E(x)}{\partial x} |\cdot\rangle = 0 \quad (2.4)$$

on physical states $|\cdot\rangle$. In these equations and in what follows, we do not show the dependence of the fields on time. Also A_1 here is the spatial component of the vector potential and E is the electric field.

As emphasized elsewhere [3, 9], the Gauss law (2.4) should be interpreted to mean

$$G(\chi)|\cdot\rangle = 0, \quad G(\chi) := \int_{-\frac{L}{2}}^{\frac{L}{2}} d\chi E \quad (2.5)$$

where the test function χ vanishes at the boundaries:

$$\chi(x)|_{x=\pm\frac{L}{2}} = 0. \quad (2.6)$$

One way to justify the formulation of the quantum Gauss law using (2.5,2.6) is as follows. A quantum field is an operator-valued distribution. Classical equations involving its derivatives should therefore be interpreted using operators obtained by smearing it with suitable derivatives of test functions. One such expression is in (2.5). But in the classical limit, it gives the classical Gauss law only if (2.6) is true. We thus justify (2.5,2.6).

Alternative and conventional justifications use classical considerations about the differentiability of functionals on the phase space [3, 9].

We can now define two operators Q_{\pm} corresponding to charges at $\pm\frac{L}{2}$ as follows:

$$\begin{aligned} Q_{\pm} &= G(\Lambda_{\pm}), \\ \Lambda_+(x)|_{\frac{L}{2}} &= 1, \quad \Lambda_+(x)|_{-\frac{L}{2}} = 0, \\ \Lambda_-(x)|_{-\frac{L}{2}} &= 1, \quad \Lambda_-(x)|_{\frac{L}{2}} = 0. \end{aligned} \quad (2.7)$$

The basis for the interpretation of Q_{\pm} as charges at $\pm\frac{L}{2}$ is discussed in [9]. Note that the action of Q_{\pm} on $|\cdot\rangle$ is independent of the value of Λ_{\pm} in the interior of our spatial interval because of (2.5). For this reason, we have not shown Λ_{\pm} as an argument of Q_{\pm} .

We emphasize that Q_{\pm} are observables localized at the edges. To see this, note that all observables localized in the interior of our interval can be constructed from $E(x)$ ($|x| < \frac{L}{2}$) and they commute with Q_{\pm} . So we can prepare states with independent choices for the values of $E(x)$ ($|x| < \frac{L}{2}$) and Q_{\pm} . In addition, changing Λ_{\pm} in the interior $|x| < \frac{L}{2}$ does not change the action of Q_{\pm} on $|\cdot\rangle$ as observed previously. For these reasons, we can think of them as edge observables.

We now show that

$$(Q_+ + Q_-)|\cdot\rangle = 0 \tag{2.8}$$

so that the charges at $\pm\frac{L}{2}$ add up to zero. The proof follows from $Q_+ + Q_- = G(\Lambda_+ + \Lambda_-)$ where $\Lambda_+ + \Lambda_-$ has the value 1 at both the edges. We can therefore choose it to be the constant function with value 1 without altering $(Q_+ + Q_-)|\cdot\rangle$. But then $Q(\Lambda_+ + \Lambda_-) = 0$ and (2.8) follows.

Next let us show that the spectrum of an edge charge is not obliged to be quantized even if the group that is gauged is $U(1)$. The mathematical reason behind this result is similar to the one leading to the possibility of fractional charges for dyons pointed out by Witten [7]. The group \mathcal{G} of gauge transformations leaving the Hamiltonian invariant are maps of $[-\frac{L}{2}, \frac{L}{2}]$ to $U(1)$. It has generators $G(\Lambda)$ with $\Lambda|_{\pm\frac{L}{2}}$ being unrestricted. Of these, the subgroup \mathcal{G}_0^B of gauge transformations generated by $G(\chi)$, with χ obeying (2.6), leaves the physical states invariant. \mathcal{G}_0^B is connected since $t\chi$ obeys (2.6) for $0 \leq t \leq 1$. [The subscript of \mathcal{G}_0^B emphasizes this fact (while B stands for boundary).] The effective symmetry group of quantum states is hence $\mathcal{G}/\mathcal{G}_0^B$. The reason charge is not quantized is because this group is not $U(1)$, but its universal cover \mathbf{R}^1 . The proof is as follows. We can choose Q_+ say as its generator, remembering (2.8). If Q_+ has a quantized spectrum, then there is a smallest period $\tau(> 0)$ such that $e^{i\tau Q_+}|\cdot\rangle = |\cdot\rangle$ or $e^{i\tau Q_+} \in \mathcal{G}_0^B$. \mathcal{G}_0^B being connected, it should therefore be possible to deform $e^{i\tau Q_+}$ to the identity staying within \mathcal{G}_0^B . But that we can not do, $e^{it\tau Q_+}$ not being identity on all states for $t < 1$ by hypothesis.

Charge is not therefore quantized.

Just as for dyons, we can understand the possibility of fractional charges from the existence of certain gauge transformations with nontrivial topology. Thus consider the group \mathcal{G}^B of all gauge transformations $e^{i\Theta}$ which act as identities at $\pm\frac{L}{2}$:

$$e^{i\Theta}|_{\pm\frac{L}{2}} = 1 \quad (2.9)$$

or

$$\Theta(-\frac{L}{2}) = 0, \quad (2.10)$$

$$\Theta(\frac{L}{2}) = 2\pi N, \quad N \in \mathbf{Z}. \quad (2.11)$$

Here (2.10) is our choice of normalization for Θ . From (2.11), one can see that \mathcal{G}^B is the disjoint union of \mathcal{G}_N^B , with Θ for \mathcal{G}_N^B fulfilling (2.10,2.11). It is only \mathcal{G}_0^B that is generated by $G(\chi)$ and is required to act as identity on physical states. If the angle Θ associated with \mathcal{G}_N^B is denoted by Θ_N , it follows that the operator T for the gauge transformation $e^{i\Theta_1}$ can have eigenvalue $e^{i\theta}$ on a physical state:

$$T|\cdot\rangle = e^{i\theta}|\cdot\rangle. \quad (2.12)$$

Note that the boundary condition (2.9) allows us to identify $x = \pm\frac{L}{2}$ and regard space as a circle S^1 when discussing \mathcal{G}^B . Thus \mathcal{G}^B consists of maps of S^1 to $U(1)$ and these are characterized by winding numbers. As these can be identified with N , we see that (2.12) is similar to the equation defining θ -states in familiar gauge theories like QCD [10, 11].

Now

$$T = e^{iG(\Theta_1)} = e^{i2\pi Q_+}. \quad (2.13)$$

It follows from (2.12) that $|\cdot\rangle$ has charge $\theta/2\pi \bmod 1$, showing that it is in general fractional. Fractional charge hence becomes possible here for topological reasons similar to those for dyons.

There is even a topological term we can include in the action which is associated with the θ -states of (2.12). It is the analogue of the integral of $\theta \text{Tr}(F \wedge F)$ in QCD [10]. It is just

$$\frac{\theta}{2\pi} \int d^2x (\partial_0 A_1 - \partial_1 A_0) \quad (2.14)$$

and is familiar from studies of the Schwinger model [12]. [Here and below, x^0 is time while x^1 is the same as the x elsewhere in this Section.] Its association to θ -states is well known and will not be repeated here.

Next consider the operator

$$W = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx A_1 . \quad (2.15)$$

It is invariant under the gauge transformations due to generators $G(\chi)$ of \mathcal{G}_0^B :

$$W \longrightarrow W + \int_{-\frac{L}{2}}^{\frac{L}{2}} d\chi = W . \quad (2.16)$$

It is hence an observable. It is conjugate to Q_{\pm} :

$$[W, Q_{\pm}] = \pm i . \quad (2.17)$$

The operator

$$V(e) = e^{ieW} \quad (2.18)$$

therefore creates charges $\pm e$ at $\pm \frac{L}{2}$:

$$Q_{\pm} V(e) |q\rangle = (q \pm e) V(e) |q\rangle \text{ if } Q_{\pm} |q\rangle = \pm q |q\rangle . \quad (2.19)$$

$V(e)$ is the analogue of the vertex operator in conformal field theories (CFT's) [13] and will be called by the same name here.

This vertex operator shifts the electric field by a constant:

$$V(e)^{-1} E(x) V(e) = E(x) + e . \quad (2.20)$$

Hence if $|0\rangle$ is the physical state with zero electric field and energy, then the energy of $V(e)|0\rangle$ is $\frac{1}{2}e^2L$, its electric field being e :

$$HV(e)|0\rangle = (\frac{1}{2}e^2L)V(e)|0\rangle . \quad (2.21)$$

In other words, the vertex operator approach gives the correct linear potential between the charges.

Since $V(e)$ does not commute with $E(x)$ for $|x| < \frac{L}{2}$, it cannot be thought of as localized at the edge. That is hence also the case for the excitations $V(e)$ creates by acting for instance on $|0\rangle$. Now as Q_{\pm} fails to commute with $V(e)$, the view that Q_{\pm} is localized at the edge requires comment. In CFT, $V(e)$ is not regarded as an observable, rather the algebra of observables is regarded as being generated by $E(x)$ and Q_{\pm} . In this point of view, which we adopt, $V(e)$ intertwines different representations of this algebra, while Q_{\pm} , which commutes with $E(x)(|x| < \frac{L}{2})$, can be thought as localized at the edge.

There is also a description of edge observables using a topological “ BF ” theory involving no metric. B here is a scalar field, we write it as E as it will later get identified with the electric field. As for F , it is just the electromagnetic field tensor dA , A being $A_0dx^0 + A_1dx^1$. The action is

$$S = \int EdA = \int dx^0dx^1[E\partial_0A_1 - E\partial_1A_0] . \quad (2.22)$$

The first term gives (2.3) while the second gives (2.4). It thus correctly reproduces the algebra of observables of the Maxwell theory. Of course, just as all topological field theories, it gives zero for the Hamiltonian.

Summarizing, observables localized at the edges are charges, the total charge of both the edges being zero and each edge charge having the option of being fractional. Wilson integrals or vertex operators can create them and also correctly reproduce their Coulomb attraction. But their kinetic energies have to be added by hand to get a complete description of their dynamics.

3 Edge Observables in 2 + 1 Dimensions

In this section, we work with a round disk D with a round hole H in the middle, so that the spatial slice is the annulus $D \setminus H$.

We begin by recollecting old results for CS and MCS actions [14, 2, 3, 4, 6]. We then move on to discuss the Higgs model. It may be noted that the material here corresponds to the case $\lambda = 0$ in ref. [4] and $\lambda = \mu = 0$ in ref. [5].

3.1 The CS Observables at the Edge

Chern-Simons Lagrangians were studied in [14, 2, 3]. They define affine Lie algebras and can be interpreted in terms of two-dimensional massless scalar fields. There are no excitations whatsoever in the interior of $D \setminus H$ in these models.

The Lagrangian and the non-vanishing equal time commutators for the $U(1)$ CS model are

$$L = \frac{k}{4\pi} \int_{D \setminus H} d^2x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (3.1)$$

$$[A_1(x), A_2(y)] = i \frac{2\pi}{k} \delta(x - y) . \quad (3.2)$$

The Gauss law operator for L is

$$\begin{aligned} G(\chi) &= \frac{k}{2\pi} \int_{D \setminus H} d\chi A , \quad \chi|_{\partial D} = \chi|_{\partial H} = 0 , \\ A &:= A_\mu dx^\mu . \end{aligned} \quad (3.3)$$

[Here and in what follows, we omit the wedge symbol between differential forms.] It annihilates physical states $|\cdot\rangle$:

$$G(\chi)|\cdot\rangle = 0 , \quad (3.4)$$

and generates a group \mathcal{G}_0^B .

The charge operators Q_{\pm} for the outer and inner edges ∂D and ∂H are given by

$$Q_{\pm} = G(\Lambda_{\pm}) , \quad (3.5)$$

$$\Lambda_{\pm}|_{|\vec{x}|=R_{\pm}} = 1 , \quad \Lambda_{\pm}|_{|\vec{x}|=R_{\mp}} = 0 , \quad (3.6)$$

∂D having the radius R_+ and ∂H the radius R_- . They commute. We can show as previously that

$$(Q_+ + Q_-)|\cdot\rangle = 0 , \quad (3.7)$$

that is that $D \setminus H$ has zero net charge.

The spectrum of Q_+ is not quantized here just as it previously was not. The proof is the same as before: $e^{i\Lambda Q_+}$ is not in \mathcal{G}_0^B for any nonzero Λ .

We will omit writing the generators of the affine Lie algebras [2, 3] at the two edges.

As in Section 2, or as for dyons, there are also nonzero winding number gauge transformations which can be regarded as the cause of charge fractionation. These transformations become identity at ∂H and ∂D and wind around $U(1)$ an integral number of times as the radial coordinate increases from a point on ∂H to a point on ∂D [15].

There is even an analogue of (2.14). If ϕ is the azimuthal angle in the annulus, it is [15]

$$\text{constant} \times \int d\phi dA. \quad (3.8)$$

The vertex operator V is essentially the parallel transport operator

$$\exp ieW \equiv \exp ie \int_L A , \quad (3.9)$$

the line L starting at a point P_- on ∂H and ending at a point P_+ on ∂D . As discussed elsewhere [3], V actually requires regularization by normal ordering so that its correct form is

$$V =: e^{ieW} : . \quad (3.10)$$

It creates charges $\pm e$ at P_{\pm} when applied to a state and is basically the Fubini-Veneziano vertex operator.

The Hamiltonian for the CS action being zero, we can not now calculate an interesting Coulomb energy for charges unlike in Section 2.

3.2 The MCS Observables at the Edge

The MCS Lagrangian is

$$\begin{aligned} L &= \int_{D \setminus H} d^2 x \mathcal{L} \\ \mathcal{L} &= -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda} . \end{aligned} \quad (3.11)$$

The Hamiltonian and Gauss law for (3.11) are

$$\begin{aligned} H &= \int_{D \setminus H} d^2 x \mathcal{H} \\ \mathcal{H} &= \frac{e^2}{2} [(\Pi_i + \frac{k}{4\pi} \epsilon_{ij} A_j)^2 + \frac{1}{e^4} (\epsilon_{ij} \partial_i A_j)^2] \\ G(\chi)|\cdot\rangle &= 0 \text{ for } \chi|_{\partial D, \partial H} = 0 , \end{aligned} \quad (3.12)$$

where

$$G(\chi) = - \int_{D \setminus H} d^2 x \partial_i \chi^{(0)} [\Pi_i - \frac{k}{4\pi} \epsilon_{ij} A_j] \quad (3.13)$$

and $|\cdot\rangle$ is any physical state.

The edge observables $Q(\Lambda_{\pm})$ for ∂D and ∂H are obtained as before from G by changing the boundary conditions on χ . They are

$$Q(\Lambda_{\pm}) = - \int_{D \setminus H} d^2 x \partial_i \Lambda_{\pm} [\Pi_i - \frac{k}{4\pi} \epsilon_{ij} A_j] , \quad \Lambda_+|_{\partial H} = \Lambda_-|_{\partial D} = 0 . \quad (3.14)$$

They are the generators of two affine Lie groups $\tilde{L}U(1)$ and have the commutators

$$[Q(\Lambda_{\epsilon}), Q(\Lambda'_{\epsilon'})] = -i \frac{k}{2\pi} \delta_{\epsilon\epsilon'} \int_{D \setminus H} d\Lambda_{\epsilon} d\Lambda'_{\epsilon'} . \quad (3.15)$$

As argued previously, since the action of $Q(\Lambda_\epsilon)$ on $|\cdot\rangle$ depends only on the boundary value of Λ_ϵ , and since it commutes with observables localized within $D\setminus H$, it can be regarded as localized at the edge.

The charges Q_\pm at ∂D and ∂H are special cases of $Q(\Lambda_\pm)$:

$$\begin{aligned} Q_+ &= Q(\Lambda_+) \quad \text{with} \quad \Lambda_+(x)|_{x \in \partial D} = 1, \\ Q_- &= Q(\Lambda_-) \quad \text{with} \quad \Lambda_-(x)|_{x \in \partial H} = 1. \end{aligned} \tag{3.16}$$

We have, as usual,

$$(Q_+ + Q_-)|\cdot\rangle = 0 \tag{3.17}$$

which means that $D\setminus H$ has zero total charge.

For $k \neq 0$, the topological field theory describing the edge states associated with (3.15) is just the CS theory, as one can readily show using the results of [2, 3].

In the present case too, just as for the *CS* case, the spectra of Q_\pm are not obliged to be quantized for reasons similar to those in Section 3.1. The topological term analogous to (2.14) is the same as in that Section, namely (3.8). Also there is a vertex operator V for the creation of charges similar to (3.10) and we can as in Section 2 calculate the Coulomb energy for the charges created by V .

The $k \rightarrow 0$ limit of the MCS system is not smooth [4]. The choice $k = 0$ gives the pure Maxwell theory, and that too can have edge observables, but with properties different from the MCS edge observables.

3.3 Edge Observables and the Higgs Model

We next summarise the work of ref [5] showing that there are no edge observables in the Higgs model.

Consider the $U(1)$ Higgs model on $D\setminus H$ with the modulus of the Higgs field frozen to its vacuum value. Its only degree of freedom left is then its phase $e^{iq\psi}$ where $q(\neq 0)$ is

the charge of the Higg's field. In this so called London limit, the Lagrangian reads

$$L = \int_{D \setminus H} d^2x \left\{ -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{m_H^2}{2} (\partial_\mu \psi - A_\mu)(\partial^\mu \psi - A^\mu) \right\}, \quad (3.18)$$

where the parameter q has been absorbed along with the vacuum value of the Higg's field (and a constant accounting for the thickness of the disk) into an effective parameter m_H (the vector meson mass being equal to em_H).

It gives rise to the Gauss law

$$G(\chi) = \int_{D \setminus H} d^2x (\chi \Pi + \frac{1}{e^2} \partial_i \chi E_i), \quad \chi|_{\partial D, \partial H} = 0, \quad (3.19)$$

annihilating the physical states $|\cdot\rangle$,

$$G(\chi)|\cdot\rangle = 0, \quad (3.20)$$

Π and E_i/e^2 being respectively the momenta conjugate to ψ and A_i .

As before, we might want to consider the observables

$$Q(\Lambda_\pm) = \int_{D \setminus H} d^2x (\Lambda_\pm \Pi + \frac{1}{e^2} \partial_i \Lambda_\pm E_i), \quad (3.21)$$

$$\Lambda_+|_{\partial H} = 0 = \Lambda_-|_{\partial D},$$

$$\Lambda_+|_{\partial D} \quad \text{and} \quad \Lambda_-|_{\partial H} \neq 0, \quad (3.22)$$

and argue that $Q(\Lambda_\pm)$ define edge states because of (3.20) and because they commute with observables localised in the interior of $D \setminus H$. If the excitations described by (3.18) correspond to the fluctuations about a vortex solution (where the singularity in ψ is wholly confined within the hole), then these edge observables will be associated with edge states associated to a vortex.

There is however a major difference between this expression for $Q(\Lambda_\pm)$ and the corresponding one in the MCS theory. In the first term of (3.21), it is the function Λ_\pm that smears the momentum Π . But we can always choose to expand a scalar function on a disc (or a disc with an annulus) in a basis of functions all of which vanish on both the

boundaries. That is, Λ_{\pm} can always be written as a (possibly infinite) sum of functions $\chi_{n_1 n_2 \dots}$, for a suitable set of indices $\{n_1 n_2 \dots\}$, where $\chi_{n_1 n_2 \dots}|_{\partial D, \partial H} = 0$. This is so despite the second line of (3.22), convergence in the expansion being in the L^2 -sense. Hence, by virtue of (3.20), $Q(\Lambda_{\pm})$ annihilates any physical state $|\cdot\rangle$ and represents the trivial observable. Edge states are thus absent in the Higgs system [at least with this treatment of $Q(\Lambda_{\pm})$].

That this is not the case in the MCS theory can be seen by noticing that the fields are smeared out by derivatives $\partial_i \Lambda_{\pm}$ of the test functions Λ_{\pm} in (3.14). Now (as we have rigorously proved in [4]) $\partial_i \Lambda_{\pm}$ can, and actually has to, be expanded as a sum of functions that do not necessarily vanish on the boundary. Hence Q_{\pm} of (3.16) do not annihilate the physical states. In fact, it is impossible to expand $d\Lambda_+$ ($d\Lambda_-$) in terms of exterior derivatives of functions that vanish at ∂D (∂H) because $d\Lambda_+$ ($d\Lambda_-$) is always orthogonal to these forms. The observables Q_{\pm} are therefore non-trivial.

4 The Boundary Conditions

In this Section, we work in 2+1 dimensions and start by stating the BC's for the MCS Lagrangian and the Maxwell and CS cases which are its limiting forms. We next do the same for the Higgs problem. From here onwards in this paper, we assume for simplicity that there is no hole so that our spatial manifold is D . There is of course no difficulty at all in restoring the hole.

4.1 The MCS Lagrangian and its Limiting Forms

Quantisation in D involves specification of BC's on ∂D . In this case, as shown elsewhere [4], there exists a one-parameter family of BC's compatible with locality and nonnegativity of the Hamiltonian for the MCS Lagrangian and its limiting forms. They are parametrised

by a number $\lambda \geq 0$. For each λ , they specify the following domain \mathcal{D}_λ for a certain Laplace like operator relevant for quantisation:

$$\mathcal{D}_\lambda = \{A | *dA|_{\partial D} = -\lambda A_\theta|_{\partial D}; \lambda \geq 0\}. \quad (4.1)$$

Here $*$ is Hodge star. Throughout this paper, we use the Hodge star only on a spatial slice and so we will need its definition only on a Euclidean manifold. The Hodge star of a p -form α on an n -dimensional Euclidean manifold with flat metric is an $n - p$ -form $*\alpha$ defined as follows:

$$(*\alpha)_{i_1 i_2 \dots i_{n-p}} = \frac{1}{p!} \epsilon_{i_1 i_2 \dots i_{n-p} j_1 j_2 \dots j_p} (\alpha)_{j_1 j_2 \dots j_p}. \quad (4.2)$$

Here ϵ is the totally antisymmetric Levi-Civita form while the components of the forms are taken along an orthonormal basis. The components of an r -form ω are themselves defined via the relation

$$\omega = \frac{1}{r!} \omega_{i_1 \dots i_r} dx^{i_1} \dots dx^{i_r} \quad (4.3)$$

where the dx^{i_k} 's are the one-forms along the orthonormal coordinate basis. Let us also note our definitions of the wedge product, and the Hodge star of the exterior derivative, in component form:

$$(\alpha^{(p)} \wedge \beta^{(q)})_{i_1 \dots i_p i_{p+1} \dots i_{p+q}} = \frac{1}{p!q!} \epsilon_{i_1 \dots i_{p+q}} \alpha_{i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}}, \quad (4.4)$$

$$(*d\alpha^{(p)})_{i_1 \dots i_{n-p-1}} = \frac{1}{p!} \epsilon_{i_1 \dots i_{n-p-1} i_{n-p} \dots i_n} \partial_{i_{n-p}} \alpha_{i_{n-p+1} \dots i_n}. \quad (4.5)$$

Here and below d on a form always refers to its exterior derivative restricted to the spatial manifold.

Also the A_θ in (4.1) has the familiar meaning:

$$A_\theta = A_i \frac{1}{r} \frac{\partial x^i}{\partial \theta}(r, \theta). \quad (4.6)$$

Here r, θ are polar coordinates on D with $r = R$ corresponding to ∂D .

Let us also note here that (as explained at length in [4]) the edge observables discussed in Section 3.2 exist only when the parameter λ defined above vanishes.

4.2 The Higgs Model in the London Limit

BC's compatible with locality and energy nonnegativity for (3.18) have been derived in [5]. They are parametrised by two nonnegative numbers λ, μ , the pair λ, μ specifying a domain $\mathcal{D}_{\lambda, \mu}$ for a certain second order operator. The definition of $\mathcal{D}_{\lambda, \mu}$ is

$$\mathcal{D}_{\lambda, \mu} = \{(\psi, A) \mid *dA|_{\partial D} = -\lambda A_\theta|_{\partial D}, \psi|_{\partial D} = -\mu(\partial_r \psi - A_r)|_{\partial D}; \lambda, \mu \in \mathbb{R}, \lambda, \mu \geq 0\}. \quad (4.7)$$

The following point is worthy of note. On D , with the magnitude of the Higgs field frozen to a constant, $e^{iq\psi}|_{\partial D}$ can have only zero winding number since $e^{iq\psi}$ will not exist as a smooth function on all of D in the contrary case. Hence $\psi|_{\partial D}$ is a well defined (single-valued) function. If this assumption is relaxed by for example restoring the hole, $e^{iq\psi}|_{\partial D}$ may have a non-zero winding number, say $N \in \mathbb{Z}$. We must then replace ψ in (4.7) by $\Delta\psi = \psi - 2\pi N/q$. This will be the appropriate scenario for the description of quantum excitations about the classical vortex solutions.

5 Physical Interpretation of Boundary Conditions

5.1 The MCS Model

We assume that D is surrounded by a superconductor with penetration depth $1/m$, and that the situation in the exterior near the edge is static, or in other words that time scales for dynamics near the edge outside the disc are long compared to typical time scales of our interest.

The order parameter for a superconductor is a complex Higgs field. It is a good approximation to take its modulus to be fixed at the vacuum value for processes involving moderate energies. Let us make this assumption and approximate the Higgs field by the

field $e^{iq\psi}$, ψ being real valued. The field equations in the exterior near ∂D that arise from the variation of (3.18) lead to

$$d * d(A - d\psi) - e^2 m_H^2 * (A - d\psi) = 0, \quad (5.1)$$

$$d * (A - d\psi) = 0, \quad (5.2)$$

(5.2) following from (5.1) on applying d . Here we have assumed that the fields are static and have used the definitions given in equations (4.2) to (4.5). Also we have not written the equations involving $A_0 - \partial_0 \psi$.

Now (5.1) and (5.2) imply that $A - d\psi$ satisfies Laplace's equation:

$$\vec{\nabla}^2(A - d\psi) = e^2 m_H^2(A - d\psi). \quad (5.3)$$

Assuming variations along azimuthal direction to be suppressed compared to radial fluctuations, this equation tells us that $A - d\psi$ decays like $e^{-em_H r}$ outside D , r being the distance from D . Thus the inverse penetration depth m is equal to em_H .

We will now examine the conditions on A and ψ near ∂D following from (5.1,5.2). Their compatibility with (4.1) will establish the claimed result that $1/\lambda$ is proportional to the penetration depth $1/m$, and will also constrain ψ .

Let P be a point on ∂D , and Q a point at a distance Δ from P in the radial direction. On integrating radially from P to Q , (5.1) gives

$$(*dA)|_Q - (*dA)|_P - m^2 \int_P^Q *(A - d\psi) = 0. \quad (5.4)$$

Now

$$\int_P^Q *(A - d\psi) = \int_P^Q \varepsilon_{ij}(A_j - \partial_j \psi) \frac{\partial x^i}{\partial r} dr = \int_P^Q (A_\theta - \frac{1}{r} \partial_\theta \psi) dr, \quad (5.5)$$

$$A_\theta - \frac{1}{r} \partial_\theta \psi = (A_i - \partial_i \psi) \frac{1}{r} \frac{\partial x^i}{\partial \theta}, \quad (5.6)$$

r and θ being polar coordinates with $r = 0$ being the center of the disc. With $m\Delta \gg 1$, the right hand side of (5.5) is approximately $(A_\theta - \frac{1}{r} \partial_\theta \psi)(P)/m$. For example, if R is

the radius of D , and we write $(A_\theta - \frac{1}{r}\partial_\theta\psi)(x) = (A_\theta - \frac{1}{r}\partial_\theta\psi)(P)e^{-m(r-R)}$, it is $\frac{1}{m}(A_\theta - \frac{1}{R}\partial_\theta\psi)|_P$. Also under the same condition, we can assume that $(*dA)_Q (= *d(A - d\psi)|_Q)$ is approximately zero. So

$$(*dA)|_P \cong -m(A_\theta - \frac{1}{R}\partial_\theta\psi)|_P. \quad (5.7)$$

Comparison with (4.1) then gives

$$\lambda = m, \quad (5.8)$$

$$\partial_\theta\psi|_{\partial D} = 0. \quad (5.9)$$

The interpretation of $1/\lambda$ in terms of the penetration depth is thus established. Equation (5.8) being in the nature of an estimate, it is best regarded as the statement of proportionality of λ and m .

Equation (5.8) requires that λ is positive. It is significant that the same sign can be derived by requiring that the Hamiltonian is non-negative.

Equation (5.9) is a condition on the static superconductor surrounding D . It is to be used as a boundary condition when solving (5.1). Although it is not invariant under gauge transformations which do not act as identity on ∂D , it is invariant under gauge transformations generated by the Gauss law, the latter becoming identity on ∂D .

As (5.2) is a consequence of (5.1), there is no need to examine it separately here.

5.2 The Higgs Model

As before, we assume that D is surrounded by a superconductor governed by the equations (5.1) and (5.2). In the interior of D too, we have a superconductor described by the fields $A, \tilde{\psi}$ (the latter decorated with a tilde to distinguish it from the phase ψ of the surrounding superconducting medium) and the boundary conditions (4.7) with $\tilde{\psi}$ replacing ψ .

Assuming the exterior of D to be static, we get (5.7) once more. Its comparison with (4.7) shows that (5.8) and (5.9) are still valid, and that $1/\lambda$ is proportional to the penetration depth in the exterior of D .

An interpretation for μ along similar lines is lacking because we do not have the analog of a “Meissner effect” for the phase of the Higgs field. Note however that if the Higgs model is to serve as an approximation for the MCS model, then the parameter μ must be zero, as will be shown in Section 6.

6 The Energy Spectrum and its Dependence on λ

We have already remarked that physical quantities are sensitive to the BC’s obeyed by the fields. In this section, we will study the energy spectrum of the MCS and Higgs theories in a more quantitative way. In particular we will study how the spectrum changes when we vary the parameter λ . [For reasons explained below we will fix μ to be zero.]

Our discussion of the MCS theory in [4] was limited by the fact that, for BC’s characterized by a nonzero value of λ , we were not able to diagonalize the Hamiltonian in (3.12) and could not therefore study its spectrum except when $\lambda = 0$.

On the other hand, in [5] we have shown that the Higgs Hamiltonian for the Lagrangian (3.18) in the London limit can be exactly diagonalized for any value of the parameters λ, μ characterizing its BC’s. We have also noticed that for $\mu = 0$, there is a one-to-one correspondence between the modes present in the Hamiltonian of the Higgs theory and the ones in the Hamiltonian of the MCS theory provided they both have the same value of λ . Indeed, both these theories describe massive vector mesons (with masses em_H and $\frac{e^2 k}{2\pi}$ respectively), the only major difference being the existence, for $\lambda = 0$, of edge states in the latter and not in the former.

We might therefore argue that the Higgs Hamiltonian will serve us as a guide in the

study of the MCS theory as well. The following results about the energy spectrum will be exact for the Higgs model for any value of λ . But they will be exact for the MCS theory only for $\lambda = 0$, that is when we are able to diagonalize its Hamiltonian exactly. For the MCS theory with $\lambda > 0$, they represent only an approximation, which is probably reasonable if λ deviates from zero only slightly.

In [5], we have second quantized the Higgs system in the London limit and shown that its Hamiltonian can be written (when $\mu = \lambda = 0$) as

$$H = \Omega_{nm}^{(\alpha)} a_{nm}^{(\alpha)\dagger} a_{nm}^{(\alpha)} + \Omega_{nm}^{(\beta)} a_{nm}^{(\beta)\dagger} a_{nm}^{(\beta)} + \Omega_n^{(h)} b_n^\dagger b_n , \quad (6.1)$$

where the $a_{nm}^{(j)}, a_{nm}^{(j)\dagger}$ ($j = \alpha, \beta$) and the b_n, b_n^\dagger are annihilation-creation operators for the three kinds of modes that appear in the mode expansion of $(A(x), \psi(x))$ when $\mu = \lambda = 0$. Their non-zero commutators are

$$\begin{aligned} \left[a_{nm}^{(j)}, a_{n'm'}^{(k)\dagger} \right] &= i\delta_{jk}\delta_{nn'}\delta_{mm'}, \\ \left[b_n, b_{n'}^\dagger \right] &= i\delta_{nn'}. \end{aligned} \quad (6.2)$$

When $\lambda \neq 0$ (but μ still equal to 0), the Hamiltonian takes an almost identical form save for the modes b_n, b_n^\dagger being absent, the latter having been deformed into a β -mode as we will see shortly.

The energies $\Omega_{nm}^{(\alpha)}, \Omega_{nm}^{(\beta)}, \Omega_n^{(h)}$ corresponding to the particle creation operators $a_{nm}^{(\alpha)\dagger}, a_{nm}^{(\beta)\dagger}, b_n^\dagger$ are respectively

$$\Omega_{nm}^{(\alpha)} = \sqrt{\alpha_{nm}^2 + e^2 m_H^2}, \quad \Omega_{nm}^{(\beta)} = \sqrt{\beta_{nm}^2 + e^2 m_H^2}, \quad \Omega_n^{(h)} = e m_H, \quad (6.3)$$

where the real numbers α_{nm}, β_{nm} are fixed by the following BC's for $\mu = 0$ and arbitrary λ :

$$\begin{aligned} J_n(\alpha_{nm}R) &= 0, \quad \alpha_{nm} \neq 0, \\ \beta_{nm} J_n(\beta_{nm}R) &= \lambda \left[\frac{d}{d(\beta_{nm}r)} J_n(\beta_{nm}r) \right]_{r=R}. \end{aligned} \quad (6.4)$$

Here, $J_n(x)$ is the real Bessel function of order n . Incidentally, we see from (6.3) that for the case $\mu = \lambda = 0$ (the case for which the modes b_n, b_n^\dagger exist), there is an infinite degeneracy of the spectrum at precisely the energy equal to the Higg's mass. Also since only the equation defining the β_{nm} depends on λ , $\Omega_{nm}^{(\alpha)}$, unlike $\Omega_{nm}^{(\beta)}$, will not change when λ is changed.

Figures 1 to 3 give plots of

$$G_n(\omega R) = \frac{\omega R J_n(\omega R)}{J'_n(\omega R)} \quad , \quad J'_n(\omega R) := \left[\frac{d}{d(\omega r)} J_n(\omega r) \right]_{r=R} \quad (6.5)$$

versus ωR for $n = 0, 1$ and 10 respectively. We can identify lines of constant λR in these figures with lines parallel to the abscissas, the ωR -coordinates of their intersections with the $G_n(\omega R)$ versus ωR giving β_{nm} . The intersections of the graphs of the functions with the abscissas at non-zero ω give instead α_{nm} .

We can now numerically calculate the dependence on λ of some of the energies $\Omega_{nm}^{(\alpha)}$, $\Omega_{nm}^{(\beta)}$ in (6.3).

Firstly, the $\alpha_{nm}R$, which are the same as $\beta_{nm}R$ when $\lambda = 0$, remain unchanged for all λ . Their values for the first few allowed m 's when $n = 0, 1$ or 10 are approximately:

$$\begin{aligned} \text{for } n = 0 & : 2.4, \quad 5.5, \quad 8.6, \dots \\ \text{for } n = 1 & : 0, \quad 3.8, \quad 7.0, \dots \\ \text{for } n = 10 & : 0, \quad 14.5, \quad 18.4, \dots \end{aligned} \quad (6.6)$$

Here $m = 1, 2$ and 3 respectively in each of the rows.

Similarly, the $\beta_{nm}R$ for λ very large (close to infinity so that the relevant points correspond to the roots of $J'_n(\omega R)$) again for the first few m 's when $n = 0, 1$ or 10 are approximately:

$$\begin{aligned} \text{for } n = 0 & : 3.8, \quad 7.0, \quad 10.2, \dots \\ \text{for } n = 1 & : 1.8, \quad 5.3, \quad 8.5, \dots \\ \text{for } n = 10 & : 11.8, \quad 16.4, \quad 20.2, \dots \end{aligned} \quad (6.7)$$

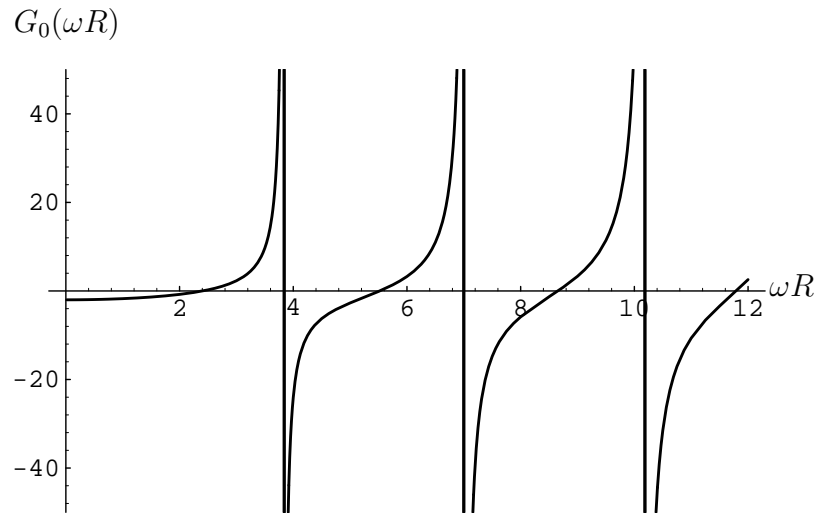


Figure 1: This figure gives the plot of $G_0(\omega R)$ vs ωR .

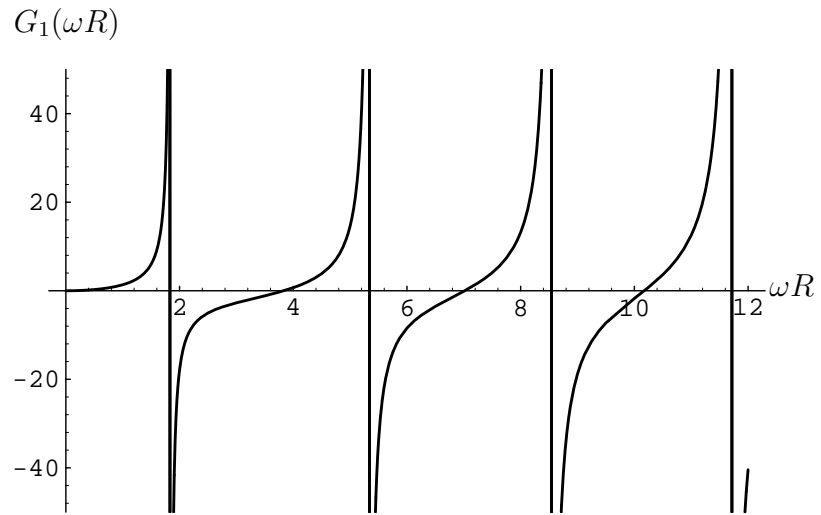


Figure 2: This figure gives the plot of $G_1(\omega R)$ vs ωR .

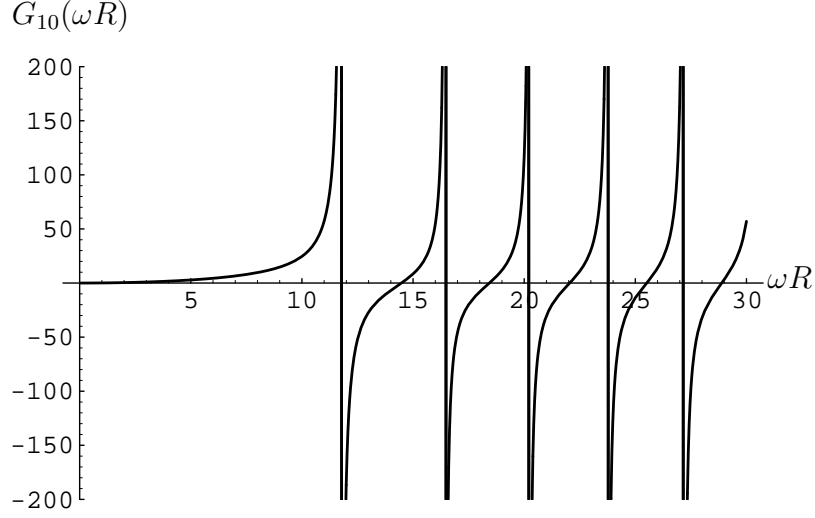


Figure 3: This figure gives the plot of $G_{10}(\omega R)$ vs ωR .

again for $m = 1, 2$ and 3 respectively in each of the rows.

In order to calculate the energies in equation (6.3), we need also to know the experimental values of R (the radius of the disc on which the above quantization has been carried out) and the mass em_H of the gauge bosons. If we further assume that $1/R$ is much smaller than em_H (which is physically reasonable being equivalent to assuming that the penetration depth is much smaller than the radius of the disc), then we can assume that $|\frac{\beta_{nm}}{em_H}| \ll 1$ (at least for the values of $\beta_{nm}R$ in (6.6) and (6.7)). Hence the following approximation for the variation $\delta\Omega_{nm}^{(\beta)}$ of $\Omega_{nm}^{(\beta)}$ under the variation $\delta\beta_{nm}$ of β_{nm} follows from $\Omega_{nm}^{(\beta)} = \sqrt{\beta_{nm}^2 + e^2 m_H^2}$:

$$\delta\Omega_{nm}^{(\beta)} = \frac{\beta_{nm}\delta\beta_{nm}}{\sqrt{\beta_{nm}^2 + e^2 m_H^2}} \approx \frac{\beta_{nm}}{em_H}\delta\beta_{nm} \quad (6.8)$$

If we choose a typical experimental value of 10^4 for $em_H R$, then the above energy dependence reduces to

$$\delta\Omega_{nm}^{(\beta)} \approx 10^{-4}\delta\beta_{nm}, \quad (6.9)$$

showing therefore that the energy changes are extremely minute (much smaller than the corresponding changes in β_{nm}) in this approximation.

However, this is no longer the case if we consider a very tiny disc whose radius is of the order of the penetration depth itself. Then instead of (6.8) we would have

$$\delta\Omega_{nm}^{(\beta)} \approx \delta\beta_{nm} \quad (6.10)$$

so that, changes in the energy are now of the same order as the changes in β_{nm} (namely, $1/R$ times the difference of the corresponding numbers in (6.6) and (6.7)).

From this rough analysis, we expect the effect of boundary conditions on the spectrum to be increasingly prominent as the size of the system under consideration becomes smaller.

7 Higher Dimensions

Edge observables and their states are consequences of gauge invariance. They can exist in gauge theories on manifolds with boundaries regardless of spatial dimensionality. The algebra of these observables is sensitive to the nature of the gauge theory and is generally abelian for an abelian gauge theory (such as the Maxwell theory) when topological interactions are absent. But it can become nonabelian even for an abelian gauge theory in the presence of the latter as happens with the CS term.

The abelian generalisation of the CS Lagrangian to 3+1 dimensions is the BF interaction involving the two-form field B and the curvature $F = dA$ of the abelian connection A . With the Maxwell term and its analogue for B included in the action, the model describes the London equations and gives a new approach to superconductivity and mass generation of vector mesons [16, 6] differing from that based on the Higgs field. We think therefore that there are good reasons to pay attention to this action. In previous work [6], we have examined it with emphasis on its edge observables and have shown that the latter are strikingly similar to their 2+1 analogues. Thus they provide a generalisation of

affine Lie groups and enjoy a description using coadjoint orbits and a topological action exactly like the WZNW action. The latter, as shown in [17], is the coadjoint action for the Kac-Moody group and is known to be associated with the edge states of the CS theory [2, 3].

This previous work on the BF system [6] had an important limitation in that it did not investigate possible BC's using operatorial methods. We have seen before [4], and will also see in Section 7.1, that BC's leading to edge observables in 2+1 dimensions are very special, and require the medium outside the manifold to be vacuum. More general BC's, involving the substitution of other media for the vacuum, delocalise the edge effects.

With this knowledge in mind, we reexamine the BF interaction in this Section with a solid ball \mathcal{B}_3 for a spatial manifold (this choice being only for specificity). The presence of parameters like those in (4.7) is established here too and their interpretation is indicated. Next we generalise our remarks of Section 3 and 4 regarding edge states of vortices to 't Hooft-Polyakov monopoles [18].

7.1 The BF Interaction

The Lagrangian of interest is

$$L = \int d^3x \mathcal{L}, \tag{7.1}$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho} - \frac{1}{3\gamma} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \\ H_{\mu\nu\lambda} &= \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}, \quad \gamma > 0, \\ \epsilon^{\mu\nu\lambda\rho} &= \text{the Levi-Civita symbol with } \epsilon^{0123} = 1. \end{aligned} \tag{7.2}$$

We first briefly review some pertinent results of [6].

(7.1) leads to the two first class constraints

$$\int_{\mathcal{B}_3} d^3x \partial_i \chi^{(0)} \Pi_i \approx 0, \tag{7.3}$$

$$-\frac{1}{2} \int_{\mathcal{B}_3} d^3x \partial_j \chi_i^{(1)} [P_{ji} + 2\epsilon_{ijk} A_k] \approx 0 \quad (7.4)$$

where $\Pi_i = F_{0i} + \epsilon_{ijk} B_{jk}$ and $P_{ij} = \frac{4}{\gamma} H_{0ij}$ are fields of momenta conjugate to A_i and B_{ij} , and $\chi^{(0)}$ and $\chi^{(1)}$ are zero and one forms with zero pull backs to $\partial\mathcal{B}_3$. We can write the latter constraints as the BC's

$$\begin{aligned} \chi^{(0)}|_{\partial\mathcal{B}_3} &= \chi^{(1)}|_{\partial\mathcal{B}_3} = 0, \\ \chi^{(1)} &\equiv \chi_i^{(1)} dx^i. \end{aligned} \quad (7.5)$$

(7.5) are found by requiring differentiability of (7.3,7.4).

The constraints (7.3,7.4) generate the gauge transformations

$$A \rightarrow A + d\chi^{(0)}, \quad (7.6)$$

$$B \rightarrow B + d\chi^{(1)}, \quad (7.7)$$

$$A \equiv A_i dx^i, \quad B \equiv \frac{1}{2} B_{ij} dx^i dx^j \quad (7.8)$$

as can be shown using the usual equal time commutators

$$\begin{aligned} [A_i(x), \Pi_j(y)] &= i\delta_{ij}\delta^3(x-y) \\ [B_{ij}(x), P_{kl}(y)] &= i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\delta^3(x-y) \end{aligned} \quad (7.9)$$

these being of course the only nonvanishing commutators for these fields. [The wedge symbol between differential forms is being omitted in (7.8) and below.]

The algebra of edge observables is generated by

$$Q(\Lambda^{(0)}) = \int_{\mathcal{B}_3} d^3x \partial_i \Lambda^{(0)} \Pi_i \quad (7.10)$$

$$R(\Lambda^{(1)}) = -\frac{1}{2} \int_{\mathcal{B}_3} d^3x \partial_j \Lambda_i^{(1)} (P_{ji} + 2\epsilon_{ijk} A_k). \quad (7.11)$$

They are got from (7.3,7.4) by relaxing the boundary conditions in (7.5) in the usual way.

Their nonvanishing commutators are all given by

$$\begin{aligned} [Q(\Lambda^{(0)}), R(\Lambda^{(1)})] &= i \int_{\mathcal{B}_3} d\Lambda^{(0)} d\Lambda^{(1)}, \\ \Lambda^{(1)} &\equiv \Lambda_i^{(1)} dx^i. \end{aligned} \quad (7.12)$$

As mentioned before, operator quantisation involves BC's on $\partial\mathcal{B}_3$ and (7.10,7.11) are well-defined only for a particular choice of these BC's. These BC's emerge when we begin diagonalising the Hamiltonian

$$\begin{aligned}
H &= \int_{\mathcal{B}_3} d^3x \left[\frac{1}{2} (\Pi_i - \epsilon_{ijk} B_{jk})^2 + \frac{\gamma}{16} P_{ij}^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{3\gamma} H_{ijk}^2 \right] \\
&= \frac{1}{2} (\Pi - 2 * B, \Pi - 2 * B) + \frac{\gamma}{8} (P, P) + \frac{1}{2} (A, *d * dA) - \frac{2}{\gamma} (B, *d * dB), \\
A &:= A_i dx^i, \quad B := \frac{1}{2} B_{ij} dx^i dx^j, \quad \Pi := \Pi_i dx^i, \quad P := \frac{1}{2} P_{ij} dx^i dx^j.
\end{aligned} \tag{7.13}$$

[As before d and $*$ refer respectively to the exterior derivative and the Hodge star on the spatial manifold (see (4.2) to (4.5) for the relevant definitions).] Here we have introduced a scalar product on the space of p -forms ($p = 1, 2$):

$$(\alpha^{(p)}, \beta^{(p)}) := \int_{\mathcal{B}_3} \overline{\alpha}^{(p)} * \beta^{(p)}. \tag{7.14}$$

Equation (7.13) shows that diagonalisation of H involves that of $*d * d$ in the space of one- and two-forms. So we must first study the domains of self-adjointness of this operator on the space of one-forms as well as of two-forms.

These domains can be found using standard rules characterising a domain, and also locality of the boundary conditions, the analysis being much the same as in [4, 5]. The possible domains $\mathcal{D}_{\lambda, \mu}$ are characterised by two parameters λ and μ and are given by

$$\mathcal{D}_{\lambda, \mu} = \{A, B | (*^{(3)} dA)|_{\partial\mathcal{B}_3} = \lambda *^{(2)}(A|_{\partial\mathcal{B}_3}); (*^{(3)} dB)|_{\partial\mathcal{B}_3} = -\mu *^{(2)}(B|_{\partial\mathcal{B}_3}); \lambda, \mu \in \mathbb{R}, \lambda, \mu \geq 0\}, \tag{7.15}$$

where the superscripts $^{(3)}$ and $^{(2)}$ on the $*$'s indicate the fact that these are defined respectively on the three-dimensional ball and its two-dimensional boundary. We shall omit these superscripts whenever there is no source for confusion about the dimension on which the $*$ is defined. The requirements $\lambda, \mu \geq 0$ arise from the requirement of non-negativity of the Hamiltonian (7.13). The quickest way to see how this comes about is by

writing the terms in H which are not manifestly positive-semidefinite as follows:

$$\frac{1}{2}(A, *d*dA) - \frac{2}{\gamma}(B, *d*dB) = \frac{1}{2}(dA, dA) + \frac{2}{\gamma}(dB, dB) - \int_{\partial\mathcal{B}} \left[\frac{1}{2}A(*^{(3)}dA) + \frac{2}{\gamma}B(*^{(3)}dB) \right]. \quad (7.16)$$

On using (7.15) in the RHS of above, we now get $\lambda, \mu \geq 0$ in order that this expression be non-negative.

The physical meaning of λ can be deduced as before by considering an arrangement with \mathcal{B}_3 surrounded by a superconductor, in general different from what is inside \mathcal{B}_3 . We can describe the latter using the Landau-Ginzburg or the BF formalism [16, 6], the latter being more appropriate for this Section, the system in \mathcal{B}_3 having been described in that way. So let A be the electromagnetic potential as before and B the two-form potential for the current J outside the ball:

$$J^\mu = -\epsilon^{\mu\nu\lambda\rho}\partial_\nu B_{\lambda\rho}. \quad (7.17)$$

The constants in the Lagrangian and field equations can now be different from (7.1,7.2) if the medium outside is different. We will therefore distinguish them by an additional prime. As in the derivation of (5.1,5.2), here too we assume a static distribution of fields in the exterior so that the time derivatives of electric/magnetic fields and of the currents are assumed to be negligible. We then get the following equations from the variation of the Lagrangian with respect to A and B :

$$-\frac{1}{3}\epsilon^{\mu\nu\lambda\rho}H_{\lambda\mu\nu} + \partial_\sigma F^{\sigma\rho} = 0, \quad (7.18)$$

$$\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} + \frac{4}{\gamma'}\partial_\rho H^{\rho\mu\nu} = 0. \quad (7.19)$$

On using also the assumption that the electric and magnetic fields and the current are static, (7.18) and (7.19) give the following equations (besides others):

$$d*dA + \frac{\gamma'}{2}P = 0, \quad x \in \mathbb{R}^3 \setminus \mathcal{B}_3, \quad (7.20)$$

$$d(A - \frac{1}{2}*P) = 0, \quad x \in \mathbb{R}^3 \setminus \mathcal{B}_3. \quad (7.21)$$

Here we have used our definitions $P_{ij} = \frac{4}{\gamma'} H_{0ij}$, $P = \frac{1}{2} P_{ij} dx^i dx^j$ and the fact that J^i is the same as $\epsilon^{ijk} H_{0jk}$ in view of (7.17).

From (7.21) it follows that $A - \frac{1}{2} * P = d\Phi$. Thus (7.20) implies the equations

$$d * dA + \gamma' * (A - d\Phi) = 0, \quad (7.22)$$

$$d * (A - d\Phi) = 0, \quad (7.23)$$

(7.23) following from (7.22) by applying d .

Exactly as in the arguments following (5.1,5.2), it can be shown that the above two equations imply that $A - d\Phi$ decays like $\exp[-\sqrt{\gamma'}(r - R)]$ outside \mathcal{B}_3 . Equation (7.22) being an equation for two-forms, we now integrate it over a two-surface S to get the BC's. The latter is chosen as follows. Let \mathcal{C}_1 be a small segment on $\partial\mathcal{B}_3$. Then S has the shape of a thin strip with \mathcal{C}_1 and another segment \mathcal{C}_2 as boundaries (see figure 4).

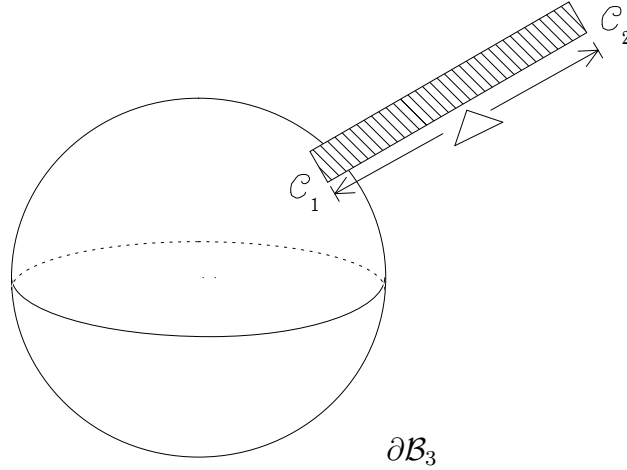


Figure 4: Figure showing a strip coming out from the boundary of ball

On integration, we find,

$$\int_{\mathcal{C}_2} *dA - \int_{\mathcal{C}_1} *dA + \gamma' \int_S *(A - d\Phi) = 0. \quad (7.24)$$

Let r, θ, φ be polar coordinates centered at the middle of \mathcal{B}_3 , $\partial\mathcal{B}_3$ having the value R and \mathcal{C}_2 the value $R + \Delta$ for r . So \mathcal{C}_2 is at a distance Δ from $\partial\mathcal{B}_3$. Choosing Δ so that

$\sqrt{\gamma'}\Delta \gg 1$, we thus find,

$$\int_{\mathcal{C}_1} *dA = \gamma' \int_S *(A - d\Phi). \quad (7.25)$$

Now, \mathcal{C}_1 being small, a further approximation is possible. If B_i is the tangential component of the magnetic field \vec{B} along \mathcal{C}_1 and $B_i(P)$ its value at a point P of \mathcal{C}_1 , (7.25) approximately gives

$$B_i(P) = -\sqrt{\gamma'}\epsilon_{ij}(A_j - \partial_j\Phi)(P) \quad (7.26)$$

Here i labels the direction of this tangent while j labels the direction perpendicular to this tangent on the surface $\partial\mathcal{B}_3$. This equation can be written as

$$(*^{(3)}dA)|_{\partial\mathcal{B}_3} = \sqrt{\gamma'} *^{(2)} [(A - d\Phi)|_{\partial\mathcal{B}_3}] \quad (7.27)$$

where the $*^{(j)}$'s here have the same meaning as in (7.15). The first minus sign in the right hand side of (7.26) is absent in (7.27) because $\partial\mathcal{B}_3$ has opposite orientations when viewed as boundary of domains outside and inside \mathcal{B}_3 . Equation (7.27) is the same as the boundary condition on A in (7.15), λ having the interpretation

$$\lambda = \sqrt{\gamma'} \quad (7.28)$$

provided also that $d\Phi|_{\partial\mathcal{B}_3} = 0$.

In other words, $\frac{1}{\lambda}$ is the penetration depth.

It merits emphasis that the sign of λ required for this interpretation is the same as the sign demanded by non-negativity of energy. [See (7.16) and what follows it.]

There remains the condition on the two-form B in (7.15). It is to be regarded as a boundary condition for solving the equations (7.20,7.21) and the other equations that arise from (7.18,7.19).

Gauss laws generate gauge transformations with test functions vanishing at $\partial\mathcal{B}_3$. The BC's (7.15) are compatible with these transformations. They are not however compatible with the transformations due to the charges (7.10,7.11) with test functions nonvanishing

at $\partial\mathcal{B}_3$ unless $\lambda = \mu = 0$. For this reason, it is only for this special choice of BC's that the edge observables exist as operators with a well-defined action on the Fock states. For other choices of BC's, they must be thought of as spontaneously broken.

The detailed quantisation of (7.1) using the BC's (7.15) is possible for $\lambda = \mu = 0$, although we have not done so. [See reference [6] in this regard.] For $\lambda, \mu \neq 0$, however, quantisation meets with the same difficulties as in the Maxwell-Chern-Simons case.

We also finally remark that a similar analysis in 3+1 dimensions can be performed using the Landau-Ginzburg, that is the gauged Higgs field, description of superconductors.

7.2 Edge States for Monopoles

The monopoles under consideration here are the 't Hooft-Polyakov monopoles in the Georgi-Glashow model [18]. In the $A_0 = 0$ gauge, they are constituted of an $SO(3)$ connection $A = A_i^\alpha dx^i \frac{1}{2} \tau_\alpha \equiv A_i dx^i$ and a Higgs field $\Phi = \varphi_\alpha \frac{1}{2} \tau_\alpha$ transforming by the $SO(3)$ triplet representation, τ_α being Pauli matrices. Let us think of the monopole as approximately confined to a ball \mathcal{B}_3 of radius R so that

$$D_i \Phi = \partial_i \Phi + ie[A_i, \Phi] = 0 \text{ for } r > R, \quad (7.29)$$

the polar coordinates r, θ, φ having origin at the center of \mathcal{B}_3 as before. The condition (7.29), as is well-known [18], is a consequence of requiring that the monopole energy is finite. For the static monopole, A and Φ become $A^{(0)} = A^{(0)3} \frac{1}{2} \tau_3$ and $\Phi^{(0)} = \Phi^{(0)3} \frac{1}{2} \tau_3$ in the U-gauge, $A^{(0)3}$ being the potential for the static Dirac monopole field and $\Phi^{(0)3}$ a nonzero constant.

Consider the field fluctuations $a_\mu, \varphi, \varphi^*$ around the static solution defined by

$$\begin{aligned} A_\mu &= [A_\mu^{(0)3} + a_\mu] \frac{1}{2} \tau_3, \\ \Phi &= \Phi^{(0)3} \frac{1}{2} \tau_3 + \varphi \frac{1}{2} \tau_+ + \varphi^* \frac{1}{2} \tau_- \end{aligned}$$

$$\begin{aligned}
&\equiv \Phi^{(0)} + \delta\Phi, \\
\tau_{\pm} &= \tau_1 \pm i\tau_2.
\end{aligned} \tag{7.30}$$

They are compatible with (7.29) if

$$\partial_i \phi + ie A_i \phi \equiv D_i \phi = 0 \text{ for } r > R. \tag{7.31}$$

Now consider the dynamics of a complex field φ with $U(1)$ charge e in interaction with a $U(1)$ connection in the region $\check{\mathcal{B}}_3$ outside the monopole. These excitations are considered with their energy localised away from the monopole.

The dynamics of a_μ and φ in $\check{\mathcal{B}}_3$ can be approximated by the usual Lagrangian L of a charged scalar field:

$$\begin{aligned}
L &= \int_{\check{\mathcal{B}}_3} d^3x \mathcal{L}, \\
\mathcal{L} &= -(D_\mu \varphi)^* (D^\mu \varphi) - \frac{1}{4} (\partial_\mu a_\nu - \partial_\nu a_\mu) (\partial^\mu a^\nu - \partial^\nu a^\mu), \\
D_\mu \varphi &\equiv \partial_\mu \varphi + ie (A_\mu^{(0)} + a_\mu) \varphi.
\end{aligned} \tag{7.32}$$

It has the Gauss law constraint

$$\begin{aligned}
\mathcal{G}(\Lambda^{(0)})|\cdot\rangle &= 0, \\
\mathcal{G}(\Lambda^{(0)}) &= \int_{\check{\mathcal{B}}_3} d^3x \Lambda^{(0)} \partial_i \Pi_i + \int_{\check{\mathcal{B}}_3} d^3x \Lambda^{(0)} ie (\varphi \pi_\varphi - \varphi^* \pi_\varphi^*), \\
\Lambda^{(0)}|_{\partial \mathcal{B}_3} &= 0 = \Lambda^{(0)}|_\infty,
\end{aligned} \tag{7.33}$$

$|\cdot\rangle$ being any physical state and Π_i and π_φ being the fields conjugate to a_i and φ . The field Π_i is of course the electric field.

The edge observables localised at the boundary $\partial \mathcal{B}_3$ of the monopole are obtained from (7.33) by partial integration:

$$Q(\Lambda) = - \int_{\check{\mathcal{B}}_3} d^3x \partial_i \Lambda \Pi_i + \int_{\check{\mathcal{B}}_3} d^3x \Lambda ie (\varphi \pi_\varphi - \varphi^* \pi_\varphi^*). \tag{7.34}$$

Here $\Lambda|_{\partial\mathcal{B}_3}$ need not vanish while $\Lambda|_\infty$ is still zero. [Nonzero $\Lambda|_\infty$ is associated with the conventional charge operator.]

The algebra of these observables is abelian:

$$[Q(\Lambda), Q(\bar{\Lambda})] = 0. \quad (7.35)$$

These observables can be interpreted as the infinitely many conserved charges of L localized at $\partial\mathcal{B}_3$.

The system in $\check{\mathcal{B}}_3$ is not in a broken phase as regards the $U(1)$ corresponding to τ_3 . We cannot therefore write $\varphi = \text{constant} (\neq 0) \times e^{i\Psi}$ in $\check{\mathcal{B}}_3$ and thereby get an approximate quadratic Lagrangian for this system. A simple gauge invariant approach for the investigation of possible BC's compatible with the Gauss law is not therefore available. This means that we are unable to generalise our previous work on BC's and display the above conditions on $\Lambda|_{\partial\mathcal{B}_3}$ as being associated with a special choice of these BC's.

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References

- [1] B.T. Halperin, Phys. Rev. B25 (1982) 2185; G. Morandi, Quantum Hall Effect, Monographs and Textbooks in Physical Science, Lecture Notes Number 10, Bibliopolis, Naples (1988); M. Stone, Ann. Phys. 207 (1991) 38; X.G. Wen, Int. Jour. Mod. Phys. B6 (1992) 1711; A.P. Balachandran and A.M. Srivastava, Syracuse University and University of Minnesota preprint SU-4228-492 TPI-MINN-91-38-T (1991) and hep-th/9111006.

- [2] E.Witten, Comm. Math. Phys. 121 (1989) 351; S.Elitzur, G.Moore, A.Schwimmer and N.Seiberg, Nucl. Phys. B326 (1989) 108 and references therein.
- [3] A.P.Balachandran, G.Bimonte, K.S.Gupta and A.Stern, Int. J. Mod. Phys. A7 (1992) 4655, 5855 and references therein.
- [4] A.P.Balachandran, L.Chandar, E.Ercolessi, T.R.Govindarajan and R.Shankar, Int. J. Mod. Phys. A9 (1994) 3417.
- [5] L.Chandar and E.Ercolessi, Syracuse University preprint SU-4240-575 (1994) and hep-th/9408047.
- [6] A.P.Balachandran and P.Teotonio-Sobrinho, Int. J. Mod. Phys. A8 (1993) 723; A9 (1994) 1569; A.P.Balachandran, G.Bimonte and P.Teotonio-Sobrinho, Mod. Phys. Lett. A8 (1993) 1305.
- [7] E.Witten, Phys. Lett. 86B (1979) 283.
- [8] R.Jackiw and C.Rebbi, Phys. Rev. D13 (1976) 3398; R.Jackiw and J.R.Schrieffer, Nucl. Phys. B190 (1981) 253.
- [9] A.P.Balachandran, “Gauge Symmetries, Topology and Quantization”, AIP conference Proceedings 317, Fifth Mexican School of Particles and Fields, Guanajuato, Mexico, 1992, Editors J.L.Lucio and M.Vargas (1994) pp. 1-81.
- [10] C.G.Callan, R.F.Dashen and D.J.Gross, Phys. Lett. 63B (1976) 334; R.Jackiw and C.Rebbi, Phys. Rev. Lett. 37 (1976) 172.
- [11] Cf. A.P.Balachandran, G.Marmo, B.S.Skagerstam and A.Stern, “Classical Topology and Quantum States”, World Scientific (1991) and references therein.
- [12] For a review, see for example N.S.Manton, Ann. Phys. (N.Y.) 159 (1985) 220.

- [13] For a review, see for example P.Goddard and D.Olive, Int. J. Mod. Phys. A1 (1986) 303.
- [14] J.F.Schonfeld, Nucl. Phys. B185 (1981) 157; S.Deser, R.Jackiw and S.Templeton, Phys. Rev. Lett. 48 (1982) 975; Ann. Phys. 140 (1982) 372.
- [15] A.P.Balachandran, G.Bimonte, K.S.Gupta and A.Stern (unpublished).
- [16] M.J.Bowick, S.B.Giddings, J.A.Harvey, G.T.Horowitz and A.Strominger, Phys. Rev. Lett. 61 (1988) 2823; J.A.Minahan and R.C.Warner, Florida Preprint UFIFT-HEP-89-15 (1989); T.J.Allen, M.Bowick and A.Lahiri, Mod. Phys. Lett. A6 (1991) 559; A.Lahiri, Mod. Phys. Lett. A8 (1993) 2403; and references therein.
- [17] B.Rai and V.G.J.Rodgers, Nucl. Phys. B341 (1990) 119; A.Yu. Alekseev and S.L.Shatashvili, Mod. Phys. Lett. A3 (1988) 1551; Nucl. Phys. B323 (1989) 719.
- [18] For a review, see for example P.Goddard and D.Olive, Rep. Prog. Phys. 41 (1978) 1357.

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